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COUNTING COLORFUL MULTI-DIMENSIONAL TREES

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Let V be a disjoint union of r finite sets $V_1,\dots,V_r$ ("colors"). A collection T of subsets of V is *colorful* if each member if T contains at most one point of each color. A *k-dimensional colorful tree* is a colorful collection T of subsets of V, each of size $k+1$, such that if we add to T all the colorful subsets of V of size k or less, we get a $\mathbb{Q}$-acyclic simplicial complex Δ_T.

We count (using the Binet–Cauchy theorem) the k-dimensional colorful trees on V (for all k), where each tree T is counted with weight $|\tilde{H}_{k-1}(\Delta_T)|^2$ ($\tilde{H}_* =$ reduced homology). The result confirms, in a way, a formula suggested by Bolker (for $k=r-1$). It extends, on one hand, a result of Kalai on weighted counting of k-dimensional trees and, on the other hand, enumeration formulas for multi-partite (1-dimensional) trees. All these results are extensions of Cayley's celebrated tree-counting formula, now 100 years old.

1. Introduction

Let $V=V_1\cup\ldots\cup V_r$ be a disjoint union of finite sets with $|V_i|=n_i$ $(1\le i\le r)$, $|V|=n=\sum_{i=1}^{r} n_i$. The elements (vertices, points) of each V_i are said to have "color" i $(1\le i\le r)$. Let Δ be the set of all subsets of V which contain at most one point of each color:

$$\Delta = \{S\subseteq V \mid (\forall i)\ |S\cap V_i|\le 1\}.$$

Δ is a pure $(r-1)$-dimensional simplicial complex, the *complete colorful complex on* V. For every integer k let $\Delta_k=\{S\in\Delta \mid |S|=k+1\}$, the set of k-dimensional simplices in Δ ($\Delta_k=\emptyset$ unless $-1\le k\le r-1$). For a subset $T\subseteq\Delta_k$ let $\Delta_T=T\cup\Delta_{k-1}\cup\ldots\cup\Delta_{-1}$. Δ_T is a simplicial subcomplex of Δ.

Definition 1.1. A *k-dimensional Δ-tree* is a subset $T\subseteq\Delta_k$ such that:
(i) $\tilde{H}_k(\Delta_T)=0$,
(ii) $\tilde{H}_{k-1}(\Delta_T)$ is a finite group.

Here $\tilde{H}_i(\Delta_T)$ denotes the i-th reduced homology group of Δ_T with integral coefficients (refer to [8] or [9] for this and other standard terms and results from algebraic topology).

For $k=1$, a 1-dimensional Δ-tree is simply a spanning tree of the 1-skeleton of Δ (i.e., the graph with vertex-set Δ_0 and edge-set Δ_1). In general, multi-dimensional Δ-trees have the following properties (to be proved later).

Proposition 1.2.
(a) *For $T\subseteq\Delta_k$, any two of the following conditions imply the third (and are equivalent to T being a Δ-tree):*

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(1) $\tilde{H}_k(\Delta_T)=0$,
(2) $\tilde{H}_{k-1}(\Delta_T)$ *is a finite group,*
(3) $|T|=\sum_{i=0}^{k}(-1)^{k-i}e_i(n_1,\dots,n_r)=\sum_{j=0}^{k}\binom{r-j-1}{r-k-1}e_j(n_1-1,\dots,n_r-1)$, *where* e_i *is the* i*-th elementary symmetric function.*

(b) *Any* $T\subseteq\Delta_k$ *satisfying* (1) *above is contained in a* Δ*-tree. Therefore* $T\subseteq\Delta_k$ *is a* Δ*-tree iff it is (inclusion) maximal among the subsets of* Δ_k *satisfying* (1).
(c) *Any* $T\subseteq\Delta_k$ *satisfying* (2) *above contains a* Δ*-tree. Therefore* $T\subseteq\Delta_k$ *is a* Δ*-tree iff it is (inclusion) minimal among the subsets of* Δ_k *satisfying* (2).

For $k=1$ this amounts to saying that a spanning tree, an (inclusion) maximal forest and an (inclusion) minimal connected spanning subgraph of Δ_1 are equivalent terms, and they have $|\Delta_0|-1$ edges.

How many k-dimensional Δ-trees are there? While discussing the (high dimensional) transportation problem, E. D. Bolker [2] wrote that "it is tempting, on numerological grounds, to guess" that there are $\prod_{i=1}^{r} n_i^{\chi_i}$ $(r-1)$-dimensional Δ-trees (here $\chi_i=\prod_{j\neq i}(n_j-1)$). He proved this result in case that $n_i\le 2$ for all except at most two values of i $(1\le i\le r)$. Bolker also suggested the formula $n^{\binom{n-2}{k}}$ for the total number of k-dimensional trees on n (labelled) vertices (in our notation: k-dimensional Δ-trees with $n_1=\ldots=n_r=1$). However, he himself noted that both formulas are false, except for very small values of the parameters.

It turns out that both formulas are indeed true, provided that we use proper *weights* while "counting" trees.

Definition 1.3. For $1\le k\le r-1$, let

$$\Sigma_k=\sum_T|\tilde{H}_{k-1}(\Delta_T)|^2,$$

where the sum extends over all k-dimensional Δ-trees.

Such a weighted counting was introduced by G. Kalai, who proved (independently of Bolker's work):

Theorem 1.4. (Kalai, 1983 [6]) *The weighted enumeration of* k*-dimensional trees in the* $(n-1)$*-dimensional simplex is*

$$\Sigma_k=n^{\binom{n-2}{k}}\qquad(1\le k\le n-1).$$

Kalai later suggested that Bolker's formula may be similarly justified for the colorful case as well. Our main result, contained in the following theorem, is that this is indeed the case. In fact, we "count" the k-dimensional Δ-trees for all values of k.

Theorem 1.5. *For* $D\subseteq\{1,\dots,r\}$ *let:*

$$d=|D|$$

$$\sigma_D = \sum_{i \notin D} n_i$$

$$\pi_D = \prod_{i \in D} (n_i - 1).$$

Then, for $1 \leq k \leq r-1$*:*

$$\Sigma_k = \prod_{d \leq k} \sigma_D^{\binom{r-2-d}{k-d}\pi_D}$$

(using the conventions: $\binom{m-1}{m} = 0$ *for* $m \geq 1$, $\binom{-1}{0} = 1$ *and an empty product* $= 1$*).*

For $k = r-1$ this indeed gives Bolker's formula:

$$\Sigma_{r-1} = \prod_{d \leq r-1} \sigma_D^{\binom{r-2-d}{r-1-d}\pi_D} = \prod_{d=r-1} \sigma_D^{\pi_D} = \prod_{i=1}^{r} n_i^{\chi_i} \qquad \left(\chi_i = \prod_{j \neq i} (n_j - 1)\right).$$

Note that the case $k = r-1$ is special in the sense that the product for Σ_{r-1} contains, eventually, only powers of σ_D for which $|D| = r-1$, whereas — for other values of k — Σ_k is a product of powers of σ_D for all D with $|D| \leq k$.

In the other extreme case ($k = 1$) — a 1-dimensional Δ-tree is simply a spanning tree (in the usual graph-theoretic sense) of the complete multi-partite graph $K_{n_1,\ldots,n_r}$. Since $\tilde{H}_0(\Delta_T)$ is always a free abelian group, it is a finite group iff it is trivial. Therefore all spanning trees appear in Σ_1 with unit weight, and we obtain from Theorem 1.5

Theorem 1.6. (Austin, 1960 [1]) *The number of (1-dimensional) spanning trees of the complete r-partite graph* $K_{n_1,\ldots,n_r}$ *(having* n_i *vertices of color i, for* $1 \leq i \leq r$*) is*

$$\Sigma_1 = n^{r-2} \prod_{i=1}^{r} (n - n_i)^{n_i - 1} \qquad \left(n = \sum_{i=1}^{r} n_i,\ r \geq 2\right).$$

In particular,

Theorem 1.7. (Fiedler and Sedlaček, 1958 [4]) *The number of spanning trees of the complete bipartite graph* K_{n_1,n_2} *is*

$$\Sigma_1 = n_1^{n_2-1} n_2^{n_1-1}.$$

As mentioned above, Kalai's result (Theorem 1.4) is the special case $n_1 = \ldots = n_r = 1$ of Theorem 1.5. It is also worth mentioning that all the results listed above are extensions of the celebrated

Theorem 1.8. (Cayley, 1889 [3]) *The number of spanning trees of the complete graph* K_n *is*

$$\Sigma_1 = n^{n-2}.$$

This paper is divided as follows: Section 2 contains a proof of Proposition 1.2, using some algebraic topology. Section 3 contains a (Δ-tree) version of the matrix tree theorem, which leads to the expression of $\Sigma_k\Sigma_{k-1}$ $(1 \le k \le r-1)$ in terms of the eigenvalues of certain matrices. Section 4 contains a computation of these eigenvalues in the special case $k = r-1$, thus evaluating $\Sigma_{r-1}\Sigma_{r-2}$. Finally, in Section 5, we compute the eigenvalues in the general case (using the case $k=r-1$ as a building block), leading to evaluation of $\Sigma_k\Sigma_{k-1}$ and Σ_k for all k. This central step is somewhat indirect, using a seemingly ad-hoc but potentially fruitful construction (see also remark (a) in Section 6). Section 6 contains some concluding remarks.

2. Colorful k-Dimensional Trees

This section contains a proof of Proposition 1.2, stating the equivalence of various definitions of a k-dimensional Δ-tree. Throughout the section Δ will denote the complete colorful complex defined in Section 1, and $1 \le k \le r-1$ will be fixed. $\tilde{\beta}_i(\Delta_T)$ will denote the i-th reduced Betti number of the simplicial complex Δ_T ($T \subseteq \Delta_k$, $0 \le i \le k$).

Notation 2.1. $t_k = \sum_{i=0}^{k} (-1)^{k-i} e_i(n_1,\dots,n_r)$, where e_i is the i-th elementary symmetric function. (By definition, $e_0(n_1,\dots,n_r)=1$.)

We need the following lemmas (to be proved in the sequel).

Lemma 2.2. $t_k = \sum_{j=0}^{k} \binom{r-j-1}{r-k-1} e_j(n_1-1,\dots,n_r-1)$.

Lemma 2.3. *For any $T \subseteq \Delta_k$, $\tilde{\beta}_k(\Delta_T) - \tilde{\beta}_{k-1}(\Delta_T) = |T| - t_k$.*

Lemma 2.4.
(a) *If $T \subseteq \Delta_k$ satisfies $\tilde{\beta}(\Delta_T)=0$ and is (inclusion) maximal among the subsets of Δ_k with that property, then T is a Δ-tree.*
(b) *If $T \subseteq \Delta_k$ satisfies $\tilde{\beta}_{k-1}(\Delta_T) = 0$ and is (inclusion) minimal among the subsets of Δ_k with that property, then T is a Δ-tree.*

Lemma 2.3 implies part (a) of Proposition 1.2, since any two of the conditions $\tilde{\beta}_k(\Delta_T)=0$, $\tilde{\beta}_{k-1}(\Delta_T)=0$ and $|T|=t_k$ imply the third. Note that $\tilde{H}_k = 0 \Leftrightarrow \tilde{\beta}_k = 0$ for a k-dimensional simplicial complex. Lemma 2.4 implies parts (b) and (c) of Proposition 1.2.

Proof of Lemma 2.2. By definition, for $1 \le i \le r$:

$$e_i(n_1,\dots,n_r) = \sum \{ n_{\alpha_1} \cdot \ldots \cdot n_{\alpha_i} \mid 1 \le \alpha_1 < \ldots < \alpha_i \le r \text{ (integers)} \}$$

and, by convention, $e_0(n_1,\dots,n_r)=1$. Writing $n_t = (n_t-1)+1$ $(t=\alpha_1,\dots,\alpha_i)$ and multiplying, we obtain: $n_{\alpha_1} \cdot \ldots \cdot n_{\alpha_i} = \sum\{(n_{\beta_1}-1)\cdot \ldots \cdot (n_{\beta_j}-1) \mid 0 \le j \le i$, and $(\beta_1,\dots,\beta_j)$ is a sub-sequence of $(\alpha_1,\dots,\alpha_i)\}$. Therefore

$$e_i(n_1,\dots,n_r) = \sum_{j=0}^{i} \binom{r-j}{i-j} e_j(n_1-1,\dots,n_r-1)$$

where $\binom{r-j}{i-j}$ is the number of sequences $(\alpha_1,\dots,\alpha_i)$ $(1\le\alpha_1<\dots<\alpha_i\le r)$ of which a given $(\beta_1,\dots,\beta_j)$ is a sub-sequence. Finally,

$$t_k=\sum_{i=0}^{k}(-1)^{k-i}e_i(n_1,\dots,n_r)=\sum_{j=0}^{k}c_j\cdot e_j(n_1-1,\dots,n_r-1)$$

where

$$c_j=\sum_{i=j}^{k}(-1)^{k-i}\binom{r-j}{i-j}=(-1)^{k-j}+\sum_{i=j+1}^{k}(-1)^{k-i}\left[\binom{r-j-1}{i-j}+\binom{r-j-1}{i-j-1}\right]$$

$$=\sum_{i=j}^{k}(-1)^{k-i}\binom{r-j-1}{i-j}+\sum_{i=j+1}^{k}(-1)^{k-i}\binom{r-j-1}{i-j-1}$$

$$=\sum_{i=j}^{k}(-1)^{k-i}\binom{r-j-1}{i-j}+\sum_{i=j}^{k-1}(-1)^{k-i-1}\binom{r-j-1}{i-j}$$

$$=\binom{r-j-1}{k-j}=\binom{r-j-1}{r-k-1}.$$ ∎

Proof of Lemma 2.3. For $T\subseteq\Delta_k$, the Euler-Poincaré formula for the (k-dimensional) simplicial complex Δ_T is:

$$\sum_{i=0}^{k}(-1)^i\tilde{\beta}_i(\Delta_T)=\sum_{i=-1}^{k}(-1)^i f_i(\Delta_T)$$

where $f_i(\Delta_T)$ is the number of i-dimensional simplices in Δ_T, i.e.:

$$f_i(\Delta_T)=\begin{cases}|\Delta_i| & -1\le i\le k-1,\\ |T| & i=k.\end{cases}$$

Now, since the $(k-1)$-skeleton of Δ_T is the same as that of Δ, we obtain $\tilde{H}_i(\Delta_T)\cong\tilde{H}_i(\Delta)$ and $\tilde{\beta}_i(\Delta_T)=\tilde{\beta}_i(\Delta)$ for $0\le i\le k-2$. The Betti numbers of Δ are well known (see, e.g., [2]):

$$\tilde{\beta}_i(\Delta)=\begin{cases}\prod_{j=1}^{r}(n_j-1) & i=r-1,\\ 0 & \text{otherwise.}\end{cases}$$

Since $k-2<r-1$, we conclude that $\tilde{\beta}_i(\Delta_T)=0$ $(0\le i\le k-2)$. Summing up:

$$(-1)^k\tilde{\beta}_k(\Delta_T)+(-1)^{k-1}\tilde{\beta}_{k-1}(\Delta_T)=(-1)^k|T|+\sum_{i=-1}^{k-1}(-1)^i|\Delta_i|.$$

Now obviously $|\Delta_i|=e_{i+1}(n_1,\dots,n_r)$ $(-1\le i\le r-1)$, and therefore

$$\tilde{\beta}_k(\Delta_T)-\tilde{\beta}_{k-1}(\Delta_T)=|T|-\sum_{i=0}^{k}(-1)^{k-i}e_i(n_1,\dots,n_r)=|T|-t_k.$$ ∎

Proof of Lemma 2.4.

(a) It is more convenient to use here homology groups with rational coefficients. Note that $\tilde{\beta}_i(\Delta_T) = \dim_{\mathbb{Q}} \tilde{H}_i(\Delta_T;\mathbb{Q})$ (as a vector-space over $\mathbb{Q}$).

For any $T \subseteq \Delta_k$, adding a k-simplex to T either increases $\tilde{\beta}_k(\Delta_T)$ by one or decreases $\tilde{\beta}_{k-1}(\Delta_T)$ by one, but not both (depending on whether the added simplex compeletes a k-cycle with rational coefficients or not). Assume now that T is (inclusion) maximal among the subsets of Δ_k satisfying $\tilde{\beta}_k(\Delta_T) = 0$. Adding any k-simplex to T increases $\tilde{\beta}_k(\Delta_T)$ and therefore does not change $\tilde{\beta}_{k-1}(\Delta_T)$. This means that the boundary of any k-simplex not in T is already in $B_{k-1}(\Delta_T;\mathbb{Q})$, the group of boundaries of k-chains supported on T. Therefore $B_{k-1}(\Delta;\mathbb{Q}) = B_{k-1}(\Delta_T;\mathbb{Q})$. Since Δ and Δ_T have the same $(k-1)$-skeleton, this implies that $\tilde{\beta}_{k-1}(\Delta_T) = \tilde{\beta}_{k-1}(\Delta) = 0$ (since $k-1 < r-1$). Hence T is a Δ-tree.

(b) The proof is similar to the previous one (deleting k-simplices from T instead of adding them). We conclude here that no simplex of T participates in any k-cycle supported on T, so that $\tilde{\beta}_k(\Delta_T) = 0$. ∎

3. The Matrix Tree Theorem for Δ-Trees

We are about to prove a Δ-tree version of the matrix tree theorem (Theorem 3.4) which will serve as the basis of our enumeration efforts. The idea to use such a result to "count" multi-dimensional trees is due to Kalai [6]. However, we shall have to take a less direct approach because of the technical complexity of our more general setting (see also remark (c) in Section 6).

Recall first a few notations: $C_k(\Delta)$, the group of k-dimensional chains of Δ, is the free abelian group generates by the elements of Δ_k. $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ is the usual boundary map. (For $k=0$ this is the augmentation map sending each vertex to $\emptyset$, the generator of $C_{-1}(\Delta)$.) $C_k(\Delta)$ has a standard basis, consisting of the elements of Δ_k (arbitrarily oriented); similarly for $C_{k-1}(\Delta)$. Let $[\partial_k]$ denote the matrix (of size $|\Delta_{k-1}| \times |\Delta_k|$) representing ∂_k with respect to these standard bases (arbitrarily ordered).

In the sequel we shall find it convenient to refer, occasionally, to Δ-trees of dimensions 0 and -1. Definition 1.1 may still be applied (where, as customary, $\tilde{H}_{-1}(\Delta_T) = 0$ for $\Delta_T \neq \{\emptyset\}$, $\tilde{H}_{-1}(\{\emptyset\}) = \mathbb{Z}$ and $\tilde{H}_{-2}(\Delta_T) = 0$ for all Δ_T). Proposition 1.2 trivially holds: A 0-dimensional Δ-tree consists of a single vertex, and a (-1)-dimensional Δ-tree is empty ($t_0 = 1$, $t_{-1} = 0$).

Lemma 3.1. *Let t_k be as in Notation 2.1. Then:*

(a) $\operatorname{rank}[\partial_k] = t_k$,

(b) $|\Delta_{k-1}| = t_k + t_{k-1}$.

Proof.

(a) $\operatorname{rank}[\partial_k]$ is the maximal number of linearly independent columns of $[\partial_k]$, i.e. the maximal size of a subset $T \subseteq \Delta_k$ s.t. $\tilde{H}_k(\Delta_T) = 0$. By proposition 1.2 ((a) and (c)) this is exactly t_k.

(b) By the definition of t_k, $t_k + t_{k-1} = e_k(n_1, \ldots, n_r) = |\Delta_{k-1}|$. ∎

For an arbitrary matrix A, with rows indexed by the elements of a set M and columns indexed by the elements of a set N, and for arbitrary subsets $I \subseteq M$, $J \subseteq N$ — let $A_{I,J}$ denote the sub-matrix of A consisting of the elements in rows indexed by I and columns indexed by J.

Theorem 3.2. *Let $0 \le k \le r-1$ and let $T_k \subseteq \Delta_k$, $\bar{T}_{k-1} \subseteq \Delta_{k-1}$ satisfy $|T_k| = |\bar{T}_{k-1}| = \operatorname{rank}[\partial_k]$. Let $T_{k-1} = \Delta_{k-1} \setminus \bar{T}_{k-1}$ and denote $d = \det([\partial_k]_{\bar{T}_{k-1}, T_k})$. Then*
(i) $d \neq 0 \Longleftrightarrow T_k$ and T_{k-1} are Δ-trees (of dimensions k and $k-1$, respectively).
(ii) If $d \neq 0$ then $d = \pm|\tilde{H}_{k-1}(\Delta_{T_k})| \cdot |\tilde{H}_{k-2}(\Delta_{T_{k-1}})|$.

Proof. The square sub-matrix $[\partial_k]_{\bar{T}_{k-1}, T_k}$ can be considered as representing the boundary map $\hat{\partial}_k : C_k(\Delta_{T_k}, \Delta_{T_{k-1}}) \to C_{k-1}(\Delta_{T_k}, \Delta_{T_{k-1}})$ of the relative complex $(\Delta_{T_k}, \Delta_{T_{k-1}})$ with respect to the standard bases of C_k and C_{k-1}. Therefore:

$$d \neq 0 \Longleftrightarrow \ker \hat{\partial}_k = 0 \Longleftrightarrow H_k(\Delta_{T_k}, \Delta_{T_{k-1}}) = 0.$$

Recall the long exact sequence of reduced homology for the pair $(\Delta_{T_k}, \Delta_{T_{k-1}})$:

$$(*) \qquad \ldots \to \tilde{H}_i(\Delta_{T_{k-1}}) \to \tilde{H}_i(\Delta_{T_k}) \to H_i(\Delta_{T_k}, \Delta_{T_{k-1}}) \to \tilde{H}_{i-1}(\Delta_{T_{k-1}}) \to \ldots$$

If $d = 0$ then $H_k(\Delta_{T_k}, \Delta_{T_{k-1}}) \neq 0$, so that $\tilde{H}_k(\Delta_{T_k})$ and $\tilde{H}_{k-1}(\Delta_{T_{k-1}})$ cannot both be zero. Therefore T_k and T_{k-1} cannot both be Δ-trees.

Assume now that $d \neq 0$. Since $\tilde{H}_k(\Delta_{T_{k-1}}) = H_k(\Delta_{T_k}, \Delta_{T_{k-1}}) = 0$ we conclude from the exactness of $(*)$ that $\tilde{H}_k(\Delta_{T_k}) = 0$. Since $|T_k| = \operatorname{rank}[\partial_k] = t_k$, we conclude by Proposition 1.2(a) that T_k is a Δ-tree. The exact sequence $(*)$ now reduces to $(**)$

$$0 \to \tilde{H}_{k-1}(\Delta_{T_{k-1}}) \to \tilde{H}_{k-1}(\Delta_{T_k}) \to H_{k-1}(\Delta_{T_k}, \Delta_{T_{k-1}}) \to \tilde{H}_{k-2}(\Delta_{T_{k-1}}) \to 0$$

(since $\tilde{H}_{k-2}(\Delta_{T_k}) = \tilde{H}_{k-2}(\Delta) = 0$).

T_k is a Δ-tree $\Rightarrow$ $\tilde{H}_{k-1}(\Delta_{T_k})$ is finite $\Rightarrow$ (by exactness of $(**)$) $\tilde{H}_{k-1}(\Delta_{T_{k-1}})$ is finite. (In fact, $\tilde{H}_{k-1}(\Delta_{T_{k-1}}) = 0$ since it is always a free abelian group.) Since $|T_{k-1}| = |\Delta_{k-1}| - t_k = t_{k-1}$ (by Lemma 3.1), we deduce from Proposition 1.2(a) that T_{k-1} is a $(k-1)$-dimensional Δ-tree. This proves (i).

Finally — recall the interpretation of $[\partial_k]_{\bar{T}_{k-1}, T_k}$ as representing the boundary map $\hat{\partial}_k : C_k(\Delta_{T_k}, \Delta_{T_{k-1}}) \to C_{k-1}(\Delta_{T_k}, \Delta_{T_{k-1}})$ of the relative complex $(\Delta_{T_k}, \Delta_{T_{k-1}})$ with respect to the standard bases of C_k and C_{k-1}. Those C_k and C_{k-1} are free abelian groups of the same rank $|T_k| = |\Delta_{k-1}| - |T_{k-1}|$, so that if $d \neq 0$ then $|d|$ is equal to the size of the quotient group $C_{k-1}/\hat{\partial}_k(C_k) = H_{k-1}(\Delta_{T_k}, \Delta_{T_{k-1}})$ ($C_{k-1} = \ker \hat{\partial}_{k-1}$ since $C_{k-2} = 0$). Using $(**)$ with $\tilde{H}_{k-1}(\Delta_{T_{k-1}}) = 0$ we obtain the exact sequence (of finite groups)

$$0 \to \tilde{H}_{k-1}(\Delta_{T_k}) \to H_{k-1}(\Delta_{T_k}, \Delta_{T_{k-1}}) \to \tilde{H}_{k-2}(\Delta_{T_{k-1}}) \to 0$$

from which we conclude that

$$|d| = |H_{k-1}(\Delta_{T_k}, \Delta_{T_{k-1}})| = |\tilde{H}_{k-1}(\Delta_{T_k})| \cdot |\tilde{H}_{k-2}(\Delta_{T_{k-1}})|.$$

Recall now

Proposition 3.3. (Binet–Cauchy Theorem [5]) *Let A and B be rectangular matrices of sizes $m \times n$ and $n \times m$ (respectively). Assume that $m \le n$. Then*

$$\det(AB) = \sum_{|I|=m} \det(A_{[m],I}) \det(B_{I,[m]}).$$

Here $[m] = \{1, \dots, m\}$ and I passes over all m-element subsets of $[n]$.

Theorem 3.4. *Let Σ_k $(1 \le k \le r-1)$ be as in Definition 1.3 and define $\Sigma_0 = n$, $\Sigma_{-1} = 1$. Then, for all $0 \le k \le r-1$, $\Sigma_k \Sigma_{k-1}$ is equal to the product of all nonzero eigenvalues of any one of the matrices $[\partial_k]^t[\partial_k]$, $[\partial_k][\partial_k]^t$ (^{t} denoting transpose).*

Proof. The product Π of all nonzero eigenvalues of a (complex) square matrix M is equal to the sum of all principal minors $\det(M_{I,I})$ with $|I|$ equal to the rank of M. Since

$$\operatorname{rank}([\partial_k]^t[\partial_k]) = \operatorname{rank}[\partial_k] = t_k$$

we get for $M = [\partial_k]^t[\partial_k]$:

$$\Pi = \sum_{\substack{T_k \subseteq \Delta_k, \\ |T_k| = t_k}} \det(([\partial_k]^t[\partial_k])_{T_k,T_k})$$

Now use Proposition 3.3 with $A = ([\partial_k]^t)_{T_k,\Delta_{k-1}}$ and $B = [\partial_k]_{\Delta_{k-1},T_k}$ to get

$$\det([\partial_k]^t[\partial_k])_{T_k,T_k} = \sum_{\substack{\bar{T}_{k-1} \subseteq \Delta_{k-1}, \\ |\bar{T}_{k-1}| = t_k}} \det(([\partial_k]^t)_{T_k,\bar{T}_{k-1}}) \det([\partial_k]_{\bar{T}_{k-1},T_k})$$

and therefore

$$\Pi = \sum_{|T_k| = |\bar{T}_{k-1}| = t_k} [\det([\partial_k]_{\bar{T}_{k-1},T_k})]^2.$$

From Theorem 3.2 we get

$$\Pi = \sum_{\substack{T_k, T_{k-1} \\ \Delta-\text{trees}}} |\tilde{H}_{k-1}(\Delta_{T_k})|^2 \cdot |\tilde{H}_{k-2}(\Delta_{T_{k-1}})|^2 = \Sigma_k \Sigma_{k-1}.$$

Finally note that $[\partial_k]^t[\partial_k]$ and $[\partial_k][\partial_k]^t$ have exactly the same *nonzero* eigenvalues (including multiplicities). ∎

Notation 3.5.

$$A_k = [\partial_k]^t[\partial_k]$$
$$B_k = [\partial_{k+1}][\partial_{k+1}]^t$$

(The subscripts are chosen so that both A_k and B_k are square matrices of the same size $|\Delta_k| \times |\Delta_k|$.)

4. The Special Case $k = r-1$

Since our initial interest (Bolker's formula) is in the values of Σ_{r-1}, it is of interest (in view of Theorem 3.4) to know the eigenvalues of A_{r-1}.

Lemma 4.1. *For $D \subseteq \{1,\ldots,r\}$, let*

$$\sigma_D = \sum_{i \notin D} n_i$$

$$\pi_D = \prod_{i \in D} (n_i - 1)$$

(with the conventions: Empty sum = 0, empty product = 1). The eigenvalues of $A_{r-1} = [\partial_{r-1}]^t[\partial_{r-1}]$ are then:

σ_D with multiplicity π_D, where D passes over all subsets of $\{1,\ldots,r\}$.

Corollary 4.2.

$$\Sigma_{r-1}\Sigma_{r-2} = \prod_{d \le r-1} \sigma_D^{\pi_D}$$

where D passes over all subsets of $\{1,\ldots,r\}$ s.t. $d = |D| \le r-1$.

(These σ_D are the nonzero eigenvalues mentioned in Theorem 3.4.)

Proof of Lemma 4.1. The entries of the matrix A_{r-1} are

$$A_{r-1}(S,S') = \begin{cases} r & S = S' \\ 1 & |S \cap S'| = r-1 \\ 0 & |S \cap S'| \le r-2 \end{cases} \qquad (S, S' \in \Delta_{r-1}).$$

Ordering the points of V according to color, we may consider Δ_{r-1} as a set of *ordered* r-tuples and identify it with $V_1 \times \ldots \times V_r$. It is then obvious that

$$A_{r-1} = \sum_{i=1}^{r} \tilde{A}^{(i)}$$

where $\tilde{A}^{(i)}$ is a 0–1 matrix having 1 only in places (S,S') such that S, S' differ (at most) in their i-th coordinate. This can be conveniently expressed by using tensor products:

$$\tilde{A}^{(i)} = I_1 \otimes \ldots \otimes I_{i-1} \otimes J_i \otimes I_{i+1} \otimes \ldots \otimes I_r \qquad (1 \le i \le r)$$

where
I_i = identity matrix of the proper size $(n_i \times n_i)$,
J_i = matrix (of size $n_i \times n_i$) with all entries equal to 1.

Now, if — for all $1 \le i \le r$ — we choose an eigenvector v_i of J_i corresponding to the eigenvalue λ_i, then $v_1 \otimes \ldots \otimes v_r$ will be an eigenvector of A_{r-1} corresponding to the eigenvalue $\lambda_1 + \ldots + \lambda_r$. Obviously, each J_i has a set of n_i linearly independent eigenvectors: One of them corresponds to the eigenvalue n_i, and the other $n_i - 1$

correspond to the eigenvalue 0. This leads to a basis of eigenvectors for A_{r-1}, of the form $\{v = v_1 \otimes \ldots \otimes v_r | v_i$ is an eigenvector of J_i $(1 \le i \le r)\}$. For each such vector v define a set $D \subseteq \{1, \ldots, r\}$ by

$$D = \{i \mid J_i v_i = 0\}.$$

Then

$$A_{r-1} v = \sum_{i=1}^{r} \tilde{A}^{(i)} v = \sum_{i \notin D} n_i v = \sigma_D v,$$

and the number of such vectors v with a given set D is

$$\prod_{i \in D} (n_i - 1) \cdot \prod_{i \notin D} 1 = \pi_D. \qquad \blacksquare$$

5. Proof of the Main Theorem

In the previous section we found the eigenvalues of A_{r-1} and could therefore compute $\Sigma_{r-1}\Sigma_{r-2}$. Unfortunately, this is not sufficient to obtain Σ_{r-1}. In order to get Σ_{r-1} itself we will have to compute $\Sigma_k \Sigma_{k-1}$ for all $1 \le k \le r-1$ and use the initial values $\Sigma_0 = |V| = n$ (a 0-dimensional tree consists of single vertex) and $\Sigma_{-1} = 1$.

There still remains a problem: We cannot compute directly the eigenvalues of $A_k = [\partial_k]^t [\partial_k]$ for general k, because of the complicated block-structure of A_k (the extreme cases $k = r-1$ and $k = 0$ are exceptionally simple). Moreover, we know that $B_{k-1} = [\partial_k][\partial_k]^t$ has the same nonzero eigenvalues as A_k — but we cannot work with B_{k-1} either (for general k) for the same reason. Our solution is to consider $A_k + B_k$.

Proof of Theorem 1.5. Let $M_k = A_k + B_k = [\partial_k]^t[\partial_k] + [\partial_{k+1}][\partial_{k+1}]^t$. For a pair of k-simplices S, $S' \in \Delta_k$, $A_k(S,S')$ is the "number" of $(k-1)$-simplices contained in $S \cap S'$ (the *common children* of S and S') and $B_k(S,S')$ is the "number" of $(k+1)$-simplices containing $S \cup S'$ (their *common parents*). "Number" here has, of course, an appropriate sign (depending on S and S').

Consider the following simple observations:

(i) If S and S' have no common children (i.e. $|S \cap S' < k$) then they also have no common parents (since then $|S \cup S'| > k+2$).

(ii) If S and S' have common children but have different color-sets (the *color-set* of S is $\{1 \le i \le r \mid S \cap V_i \ne \emptyset\}$) — then S and S' have a *unique* common child and also (necessarily $k \ne r-1$) a unique common parent. Moreover, $A_k(S,S')$ and $B_k(S,S')$ have *different* signs, so that $M_k(S,S') = 0$.

We have seen so far that $M_k(S,S') = 0$ unless S and S' have the same color-set. Therefore M_k is a direct sum of its square sub-matrices $M_k(C)$ corresponding to the various color-sets C:

$$M_k = \bigoplus_{C \in \binom{[r]}{k+1}} M_k(C)$$

where

$$\binom{[r]}{k+1} = \{C \mid C \subseteq [r] \equiv \{1, \ldots, r\},\ |C| = k+1\}$$

and $M_k(C)$ is the sub-matrix of M_k containing only rows and columns corresponding to the $S \in \Delta_k$ with color-set C.

Now, every matrix $M_k(C) = A_k(C) + B_k(C)$ (obvious definitions) can be analyzed as follows:

(i) $A_k(C)$ is the matrix of common children ($[\partial_k]^t[\partial_k]$) for the k-dimensional simplices in the k-dimensional complete colorful complex with color-partition $\{V_i \mid i \in C\}$. Therefore Lemma 4.1 applies and the eigenvalues of $A_k(C)$ are:

$$\sigma'_D = \sum_{i \in C \setminus D} n_i, \text{ with multiplicity } \pi_D = \prod_{i \in D} (n_i - 1) \quad (\forall D \subseteq C).$$

(ii) $B_k(C) = \sigma_C I$ is a scalar matrix ($\sigma_C = \sum_{i \notin C} n_i$), since different simplices with the same color-set have no common parent.

Summing up, the eigenvalues of $M_k(C)$ are:

$$\sigma_D = \sigma'_D + \sigma_C = \sum_{i \notin D} n_i, \text{ with multiplicity } \pi_D = \prod_{i \in D} (n_i - 1) \quad (\forall D \subseteq C).$$

Since, for every $D \subseteq [r]$ with $|D| = d$,

$$\left| \left\{ C \in \binom{[r]}{k+1} | D \subseteq C \right\} \right| = \binom{r-d}{k+1-d}$$

we obtain

Lemma 5.1. *The eigenvalues of $M_k = A_k + B_k$ are:*

$$\sigma_D = \sum_{i \notin D} n_i, \text{ with multiplicity } \binom{r-d}{k+1-d} \pi_D = \binom{r-d}{k+1-d} \prod_{i \in D} (n_i - 1),$$

where D may be any subset of $[r]$ satisfying $d = |D| \leq k+1$.

We still have to find the eigenvalues of A_k (or B_k).

Lemma 5.2.

$$A_k B_k = B_k A_k = 0.$$

Proof. Follows immediately from the definitions of A_k and B_k and from the fundamental property $[\partial_k][\partial_{k+1}] = [\partial_k \partial_{k+1}] = 0$. ∎

Lemma 5.3.

(i) *The multiset of nonzero eigenvalues of M_k is the (multiset) union of the multisets of nonzero eigenvalues of A_k and B_k.*

(ii)

$$\Sigma_{k+1} \Sigma_k^2 \Sigma_{k-1} = \prod_{d \leq k+1} \sigma_D^{\binom{r-d}{k+1-d} \pi_D} \qquad (0 \leq k \leq r-2),$$

where the product is over all $D \subseteq [r]$ with $d = |D| \leq k+1$.

Proof.

(i) A_k and B_k are commuting real symmetric matrices (by Lemma 5.2), and therefore there is a basis $\{v_1, \ldots, v_N\}$ of common eigenvectors ($N = |\Delta_k|$). Let α_i, β_i

and μ_i be the eigenvalues of A_k, B_k and M_k (respectively) corresponding to the eigenvector v_i.

$$A_k B_k = 0 \Rightarrow (\forall i)\alpha_i\beta_i = 0 \Rightarrow (\forall i)(\alpha_i = 0 \text{ or } \beta_i = 0),$$

so that $\mu_i = \alpha_i + \beta_i = \begin{cases} \beta_i & \alpha_i = 0,\ \beta_i \neq 0 \\ \alpha_i & \alpha_i \neq 0,\ \beta_i = 0 \\ 0 & \alpha_i = \beta_i = 0 \end{cases}$

This proves (i).

(ii) This follows from (i), Lemma 5.1 and the fact (Theorem 3.4) that the product of all nonzero eigenvalues of $A_k(B_k)$ is $\Sigma_k\Sigma_{k-1}$ ($\Sigma_{k+1}\Sigma_k$, respectively). ∎

To complete the proof of Theorem 1.5 note that Lemma 5.3(ii) determines all Σ_k, given the values of Σ_{-1} and Σ_0. The expressions for Σ_k in Theorem 1.5, together with $\Sigma_{-1} = 1$ and $\Sigma_0 = n$, satisfy the equations of Lemma 5.3(ii) and are therefore correct for all values of k. (Note that the formula in Theorem 1.5, although stated only for $1 \leq k \leq r-1$, gives the correct values $\Sigma_{-1}=1$, $\Sigma_0=n$ and $\Sigma_k=0$ for $k \geq r$.)

6. Concluding Remarks

(a) One should note the central role played (in the proof of the main theorem) by the matrix

$$M_k = A_k + B_k = [\partial_k]^t[\partial_k] + [\partial_{k+1}][\partial_{k+1}]^t$$

which represents the Laplacian operator $\delta_{k-1}\partial_k + \partial_{k+1}\delta_k$ on $C_k(\Delta)$. (Here $\partial_k : C_k \to C_{k-1}$ and $\delta_k : C_k \to C_{k+1}$ are the boundary and coboundary maps on C_k respectively.) In our case M_k could be conveniently expressed as a direct sum, thus facilitating further computations. This phenomenon might occur in other situations as well. For example, let $\Delta_1, \ldots, \Delta_r$ be simplicial complexes on disjoint vertex-sets $V_1, \ldots, V_r$. Their *free join* $\Delta = \Delta_1 * \ldots * \Delta_r$ is the simplicial complex on the vertex-set $V = V_1 \cup \ldots \cup V_r$ defined by:

$$\Delta = \{S_1 \cup \ldots \cup S_r \mid (\forall i) S_i \in \Delta_i\}.$$

Note that if $\Delta_1, \ldots, \Delta_r$ are *0-dimensional* (i.e., composed of isolated points only) then Δ is the full colorful complex being dealt with so far. In any case, $M_k(\Delta)$ may be expressed as a direct sum of sums of tensor products (like those appearing in the proof of Lemma 4.1, but with suitable $M_j(\Delta_i)$ instead of $J_i = A_0(\Delta_i)$).

(b) More elaborate versions of Theorem 1.5 may be proved by suitable modifications of the method outlined in this paper. One may consider, for example, the enumeration of k-dimensional Δ-trees according to their sequences of vertex-degrees (described in [6] for the "uncolored" case $n_1 = \ldots = n_r = 1$). See also [7, p.41] for similar results concerning 1-dimensional trees.

(c) Our initial interest was in Bolker's formula for Σ_{r-1}. The problem was that even having found all the eigenvalues of A_{r-1} (which we did in Section 4) we only knew the product $\Sigma_{r-1}\Sigma_{r-2}$; and in order to recover Σ_{r-1} itself we had to deal with A_k (and B_k) for *all* values of k.

An alternative way is to find a matrix which has Σ_{r-1} as the product of its nonzero eigenvalues. By the arguments of Section 3, this may be achieved if we replace $A_{r-1} = [\partial_{r-1}]^t[\partial_{r-1}]$ by $\tilde{A}_{r-1} = D^tD$, where $D = [\partial_{r-1}]_{T,\Delta_{r-1}}$ is the sub-matrix of $[\partial_{r-1}]$ with rows corresponding to the $(r-2)$-simplices in some *fixed* $(r-2)$-dimensional Δ-tree T (say with $\tilde{H}_{r-3}(\Delta_T) = 0$). This is, in fact, the method used in [6] to deal with the case $n_1 = \ldots = n_r = 1$. For the reader who may wish to pursue this route we offer the following natural candidate for a k-dimensional Δ-tree T (with $\tilde{H}_{k-1}(\Delta_T) = 0$), suggested by the formula in Lemma 2.2:

Choose an element $x_i \in V_i$ ("head") for each color $1 \le i \le r$. To each colorful subset S of V ($|S \cap V_i| \le 1$ for all i) assign a signature $\chi_S = (\varepsilon_1, \ldots, \varepsilon_r) \in \{1, 0, -1\}^r$ by:

$$\varepsilon_i = \begin{cases} 1 & S \cap V_i \text{ consists of the head of } V_i \\ -1 & S \cap V_i \text{ consists of a non-head of } V_i \\ 0 & S \cap V_i = \emptyset. \end{cases}$$

The Δ-tree T then consists of all colorful $S \subseteq V$ with signature $\chi_S = (\varepsilon_1, \ldots, \varepsilon_r)$ satisfying:

(i) $\sum_{i=1}^{r} |\varepsilon_i| = k+1$ (S is k-dimensional).

(ii) There is an index $1 \le i_0 \le r$ s.t. $\varepsilon_{i_0} = 1$ and $\varepsilon_i = -1$ for $i < i_0$.

Different such Δ-trees may be obtained by changing the order of the r colors. In the special case $n_1 = \ldots = n_r = 1$, this tree consists of all the k-dimensional faces of the $(n-1)$-dimensional simplex which contain a fixed vertex (the unique element of V_1).

(d) (Due to A. Frumkin.) The computation of the eigenvalues of A_k (and B_k) in the special case $n_1 = \ldots = n_r = 1$ (leading to Theorem 1.4) may be accomplished also through the representation theory of the symmetric group S_n, as follows: The natural action of S_n on the set V of vertices of Δ induces an action of S_n on each rational chain group $C_k(\Delta; \mathbb{Q})$, which may be viewed as the exterior power $\wedge^{k+1}\mathbb{Q}^n$. It is well-known that, as a $\mathbb{Q}S_n$-module, $\wedge^{k+1}\mathbb{Q}^n \cong M_k \oplus M_{k+1}$ where M_k is the irreducible $\mathbb{Q}S_n$-module corresponding to the hook of (size n and) height $k+1$ ($\dim_{\mathbb{Q}} M_k = \binom{n-1}{k}$). The boundary map ∂_k and the coboundary map δ_{k-1} are obviously homomorphisms of $\mathbb{Q}S_n$-modules:

$$M_k \oplus M_{k+1} \xrightarrow{\partial_k} M_{k-1} \oplus M_k \xrightarrow{\delta_{k-1}} M_k \oplus M_{k+1}.$$

The matrix $A_k = [\partial_k]^t[\partial_k] = [\delta_{k-1}\partial_k]$ represents their composition. By Schur's lemma one gets $\partial_k\Big|_{M_{k+1}} = 0$, $\partial_k\Big|_{M_k} = \alpha_k I$ (for a suitable constant α_k) and similarly $\delta_{k-1}\Big|_{M_{k-1}} = 0$, $\delta_{k-1}\Big|_{M_k} = \beta_k I$. Therefore $\delta_{k-1}\partial_k = \alpha_k\beta_k I\Big|_{M_k} \oplus 0\Big|_{M_{k+1}}$ has only one nonzero eigenvalue:

$$\alpha_k\beta_k = \frac{\operatorname{tr} A_k}{\dim M_k} = \frac{\binom{n}{k+1}(k+1)}{\binom{n-1}{k}} = n,$$

with multiplicity $\dim M_k = \binom{n-1}{k}$.

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